One-loop Effective Action in Quantum Gravitation

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Abstract We present the formalism of computing one-loop effective action for Quantum Gravitation using non-local heat kernel methods. We found agreement with previous old results. In main part of my presentation I considered the system of E-H gravitation and scalar fields. We were able to derive nonlocal quantum effective action up to the second order in heat kernel generalized curvatures. By going to flat spacetime expressions for gravitational formfactors are possible to construct and compare with the results from effective field theory for gravity.

1 Truncation ansatz and 'inverse propagator'

In this work we will review the results of computation of 1-loop effective action in a system, where we have standard Einstein-Hilbert gravitation and a minimally coupled scalar field. Standard computation, known in the literature, are mainly based on perturbative quantization methods and they exploit Feynman diagrams techniques [3]. Here we will follow a different route. Namely we will obtain 1-loop quantum effective action as the effect of integrating average effective action along the RG flow trajectory from UV down to IR limit. Moreover in the core of our calculation we will use non-local heat kernel techniques to evaluate some functional traces. We will pay special attention to the appearance of nonlocal terms in the quantum effective action. All the calculations will be performed in Euclidean spacetimes and later we will specify to four spacetime dimensions. One of the goal of such calculation

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is the quantum effective action per se. Another is related to gravitational formfactors of simple interactions with scalars.

Now we want to introduce the notion of the average effective action (EAA). The EAA is a scale-dependent generalisation of the standard effective action, that interpolates smoothly between the bare action for $k \to \infty$ and the standard quantum effective action for $k \to 0$. In this way, we avoid the problems of performing the functional integral. Instead they are converted into the problem of integrating the exact flow of the EAA from the UV to the IR. The EAA formalism deals naturally with several different aspects of quantum field theories. One aspect is related to the discovery of non-Gaussian fixed points of the RG flow. In particular, the EAA framework is a useful setting to search for Asymptotically Safe theories, i.e. theories valid up to arbitrarily high energy scales. A second aspect, in which the EAA reveals its big usefulness, is the domain of nonperturbative calculations. In fact, the exact flow, that EAA satisfies is a valuable starting point for inventing new approximation schemes.

In EAA the crucial point is the separation between high and small energy modes of quantum fields. The elimination of higher energy modes is performed by separating the low energy modes, to be integrated out, from the high modes in a covariant way. To do this we introduce a cutoff action constructed using the covariant d'Alambertian, that respects the symmetries of the underlying theory. In full generality in order to construct EAA we add to the bare action S an infrared (IR) "cutoff" or regulator term ΔS_k of the form:

$$\Delta S_k = \frac{1}{2} \int \mathrm{d}^d x \sqrt{g}\, \phi R_k(\Box) \phi\,. \tag{1}$$

In above formula the operator kernel R_k is chosen in such a way to suppress the field modes ϕ_n, eigenfunctions of the covariant second differential operator $\Box$, with eigenvalues smaller than the cutoff scale $v_n < k^2$. Generic fields of our quantum field theory are denoted here by ϕ. We will call ΔS_k the cutoff action. The functional form of the cutoff kernels $R_k(z)$ is arbitrary except for the requirements that they should be monotonically decreasing functions in both z and k arguments, i.e. rigorously that $R_k(z) \to 0$ for $z \gg k^2$ and that $R_k(z) \to k^2$ for $z \ll k^2$. It is important to recall two limits of EAA. First in the IR limit ($k = 0$) quantum effective action is obtained. On the other hand, when $k \to \infty$, then EAA equals to the bare action of considered quantum theory. In this way we obtain the scale dependent generalisation of the standard effective action, which interpolates between the two.

Quantum gravity gives unambiguous predictions at low energy in the framework of effective field theories. The low energetic action contains only the simplest Einstein-Hilbert term (with a possibility of adding a cosmological constant, which we however neglect here). In this effective theory there exist observables, which do not depend on the particular way of UV completion. They are genuine predictions of quantum gravity. The quantum divergences, which must be absorbed during the renormalization procedure, are contained in local, but not universal terms in the quantum effective action. We are mainly interested in nonlocal terms in quantum effective action. The reason for this is that they are universal terms in low-energetic effective field theory of quantum gravity [4, 5]. They do not depend on any specific way of UV completion of gravity. There exist different ways, by which, one can obtain quantum effective action in the infrared limit. However it is without any doubt that low-energetic predictions of quantum gravity are calculable and solid, regardless of any complicated dynamics, which saves the theory in UV. In our method for integration RG flows we will use exact (also known as functional) Renormalization Group equations. In integration of RG flow of scale-dependent effective action such nonlocal terms originate from the part of integration done for the lowest momentum scales.

We will use the following ansatz for the form of the action of our system

$$
\begin{aligned}
S = & \int \mathrm{d}^d x \sqrt{g} \left[\frac{1}{K^2} R - \frac{1}{2} (\partial\phi)^2 - V(\phi) \right] - \frac{1}{2K^2\alpha} \int \mathrm{d}^d x \sqrt{g}\, \chi^2 \\
& + \int \mathrm{d}^d x \sqrt{g}\, \bar{C}_\mu \left(-\Box \delta^\mu_\nu - R^\mu_\nu \right) C^\nu , \qquad (2)
\end{aligned}
$$

where d'Alambertian is given by $\Box = \nabla_\mu \nabla^\mu$. Due to the gauge diffeomorphism symmetry present in the system we are forced to introduce gauge fixing conditions necessary for perturbative quantization of the system: $\chi_\mu = \nabla^\nu h_{\mu\nu} - \frac{1}{2}\nabla_\mu h$. Moreover another consequence of this gauge redundancy is that for consistency, we also had to add vector ghosts denoted by C_μ in the second line of (2). In our computation we use the background field method and we take the metric perturbations in the form $h_{\mu\nu} = \delta g_{\mu\nu}$ and in contracted version $h = g^{\mu\nu} h_{\mu\nu}$. All covariant derivatives are with respect to the background metric. As we can see in the action (2) we included minimally coupled scalar field ϕ and we allow for the existence of potential $V(\phi)$ for it. Gravitational coupling appears there as K, which has the inverse energy dimension. In the gravitational part of the action R is the only present curvature invariant built out of the full metric $g_{\mu\nu}$. Additionally constant α is a gauge parameter in our gauge fixing condition.

When we have the explicit form of the action, then the next step is to compute the second variational derivative w.r. to all fluctuating quantum fields like in [1]. Usually this takes the form of second order differential operator, which is of fundamental importance in our construction of the cutoff kernels in the EAA.

2 Exact RG flows

Using the methods of nonlocal heat kernel we will now exploit the power of Exact RG formalism applied to the EAA. At the beginning we need to know flows of which terms to consider and for this reason we first look for simple task related to local terms.

2.1 Local terms of one-loop effective action

Firstly we will look for local terms in 1-loop effective action for our system. They are related to UV divergences of the theory. In general these divergences give rise to the renormalization of couplings in front of local terms. They are not universal and depend on the precise way of UV completion. However we assume, that the bare action (in UV) is given by (2). At one loop order the quantum effective action is given by the integral

$$\Gamma[\phi, g] = -\frac{1}{2}\int_0^\infty \frac{\mathrm{d}s}{s} \mathrm{Tr}\, \mathrm{e}^{-s\hat{S}^{(2)}}\,, \tag{3}$$

where $\mathrm{Tr}\, \mathrm{e}^{-s\hat{S}^{(2)}}$ is the functional trace of some differential operator, which we are going to compute with the heat kernel techniques. For our applications in the exponent of heat kernel we use inverse propagator, spoken about in the previous section, denoted here by $\hat{S}^{(2)}$ (second variational derivative of the action S with respect to all fluctuating fields). This operator, as other quantities with a hat over, is a matrix in field space of gravitons and scalar field perturbations. In order to find logarithmically divergent part of one-loop effective action to second order in curvature we can use the Schwinger-DeWitt method for quadratic operators:

$$\begin{aligned}\mathrm{Tr}\, \mathrm{e}^{-s\hat{S}^{(2)}} = \frac{1}{(4\pi s)^{d/2}} \int \mathrm{d}^d x\, \sqrt{g}\, \mathrm{tr} \Big\{ \hat{1} + s\hat{P} + s^2 \Big[\frac{1}{2}\hat{P}^2 + \frac{1}{12}\hat{\mathcal{R}}_{\mu\nu}\hat{\mathcal{R}}^{\mu\nu} + \\ + \frac{1}{180}\mathrm{Riem}^2 \hat{1} - \frac{1}{180} R_{\mu\nu}R^{\mu\nu}\hat{1} \Big] \Big\}\,. \end{aligned} \tag{4}$$

We will restrict ourselves to second order contribution in operators $\hat{P}$, $\hat{\mathcal{R}}_{\mu\nu}$ and gravitational curvatures. (We don't consider here application of this method to the ghost part of the action, because we are mainly interested in terms with nonminimally coupled matter.) Using Schwinger-DeWitt technique we reduced the functional trace to matrix traces. The terms, which appear explicitly in the above expression are the basis for consideration of RG flows for nonlocal operators.

2.2 Nonlocal terms and their exact RG flows

In order to go beyond Schwinger-DeWitt technique and find form of nonlocal part of one-loop action we insert nonlocal structure functions. They are functions of s parameter and box operator $\Box = \nabla^\mu \nabla_\mu$ (acting under the integral). We insert these structure functions between two matrix operators present at the second order as in the detailed formula below

$$\frac{1}{(4\pi s)^{d/2}} \int \mathrm{d}^d x \sqrt{g}\, s^2 \,\mathrm{tr} \Big\{ \Big[\hat{P}\, f_P\,(-s\Box)\, \hat{P} + \hat{\mathcal{R}}_{\mu\nu} f_{\mathcal{R}}\,(-s\Box)\, \hat{\mathcal{R}}^{\mu\nu} + \\ + \hat{P}\, f_{PR}\,(-s\Box)\, R + R\, f_R\,(-s\Box)\, R\, \hat{1} + R_{\mu\nu} f_{\mathrm{Ric}}\,(-s\Box)\, R^{\mu\nu}\, \hat{1} \Big] \Big\} \,. \qquad (5)$$

It must be emphasised, that the leading order in s contribution is equal to constants, which were written in the formula (4) in section above (for $\hat{P}R$ operator this constant vanishes). Moreover we have used the Euler identity here. The traces of matrix terms of order curvature square are modified with respect to expressions given in previous section by the appearance of structure functions f_P, $f_{\mathcal{R}}$, f_{PR}, f_R and f_{Ric}.

Now we want to consider the exact RG flow of EAA, which will be denoted here by $\bar{\Gamma}_k$. As the ansatz for it we choose the nonlocal expression above, understood that all the couplings and structure functions now acquire dependence on the momentum scale k. The exact RG flow equation for the background effective average action (bEAA) is the following

$$\partial_t \bar{\Gamma}_k[\phi, g] = \frac{1}{2} \mathrm{Tr} \frac{\partial_t R_k(-\mathcal{D}^2))}{-\mathcal{D}^2 + R_k(-\mathcal{D}^2)} - \mathrm{Tr} \frac{\partial_t R_k(\Delta_{gh})}{\Delta_{gh} + R_k(\Delta_{gh})} \,. \qquad (6)$$

In the above formula $\mathcal{D}$ is a general operator of the covariant derivative and R_k are cutoff kernels (suitably chosen functions of momenta to suppress the contributions from high energy modes in the path integral). The r.h.s. of this equation expresses itself by functional traces of some differential operators and the RG time derivatives of cutoff kernels $t = \log k/k_0$. We note that in the denominator we have differential part $\mathcal{D}^2$ of our inverse propagator operator. The r.h.s. of the flow equation is then schematically given as

$$\partial_t \bar{\Gamma}_k[\phi, g] = \frac{1}{(4\pi)^{d/2}} \int \mathrm{d}^d x \sqrt{g} \left\{ \mathcal{O}_{i,1} \left[\int_0^\infty \mathrm{d}s\, \tilde{h}_k(s)\, s^{2-\frac{d}{2}}\, \tilde{f}_i(s\Box) \right] \mathcal{O}_{i,2} \right\} \,, \qquad (7)$$

where the structure functions $\tilde{f}_i(x)$ were derived combining non-local heat kernel structure functions and $\mathcal{O}_{i,1,2}$ stand for operators in between which we insert these structure functions. In this derivation we follow [6].

3 Effective action and formfactors

Finally integrating the flow (7) and putting some boundary conditions in UV, we arrive to the following explicit form for one-loop quantum effective action in our model of scalar field interacting minimally with Quantum Gravitation:

$$
\begin{aligned}
\bar{\Gamma}_0\big|_{\mathcal{R}^2} = \frac{1}{32\pi^2}\int \mathrm{d}^4x\sqrt{g}\,\Bigg\{ & \frac{71}{30}R_{\mu\nu}\log\left(\frac{-\Box}{k_0^2}\right)R^{\mu\nu} + \frac{71}{60}R\log\left(\frac{-\Box}{k_0^2}\right)R + \\
+ & \frac{5}{2}K^4m^4\phi^2\log\left(\frac{-\Box}{k_0^2}\right)\phi^2 - 2K^2m^4\phi\log\left(\frac{-\Box}{k_0^2}\right)\phi \\
- & \frac{13}{3}K^2m^2R\log\left(\frac{-\Box}{k_0^2}\right)\phi^2 - \frac{1}{6}m^2R + \frac{1}{2}m^4 + \frac{5}{2}K^4\left(\nabla\phi\right)^2\log\left(\frac{-\Box}{k_0^2}\right)\left(\nabla\phi\right)^2 \\
+ & K^4m^2\phi^2\log\left(\frac{-\Box}{k_0^2}\right)\left(\nabla\phi\right)^2 - \frac{2}{3}K^2R\log\left(\frac{-\Box}{k_0^2}\right)\left(\nabla\phi\right)^2 - K^2m^2\left(\nabla\phi\right)^2\Bigg\}\,.
\end{aligned} \tag{8}
$$

This 1-loop quantum effective action is the main, solid result of this work.

The goal of this section would be to compute one-loop corrections to three-point vertex from above action. Above we have computed it to the second order in operators of heat kernel and we arrived at a nonanalytic expression with low-energetic logarithms. We want to consider the simplest vertex of interaction within our theory — with one gravitons and two scalar fields. That's why we shall compute the third variational derivative with respect to mentioned fluctuations. At the end we specify flat gravitational background and vanishing background scalar field. We also prefer to write the expression for the vertex in the momentum space and in such way we can compare to the perturbative results. Such comparison and more details of this computation and these techniques can be found in [2]. However in this short contribution we shed light only on the most important aspects of the lengthy calculations.

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References

1. A. O. Barvinsky, A. Y. .Kamenshchik and I. P. Karmazin, Phys. Rev. D **48**, 3677 (1993) [gr-qc/9302007].
2. A. Codello, R. Percacci, L. Rachwał and A. Tonero, in preparation
3. J. F. Donoghue, In *Glasgow 1994, Proceedings, High energy physics, vol. 2* 1061-1063, and Massachusetts U. Amherst - UMHEP-413 (rec.Oct.94) 6 p [hep-th/9409143].
4. J. F. Donoghue, gr-qc/9512024.
5. J. F. Donoghue, AIP Conf. Proc. **1483**, 73 (2012) [arXiv:1209.3511 [gr-qc]].
6. A. Satz, A. Codello and F. D. Mazzitelli, Phys. Rev. D **82**, 084011 (2010) [arXiv:1006.3808 [hep-th]].